

# Extracting $V_{\text{us}}$ from Lattice QCD simulations: Recent progress and prospects

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I review the current status of the determination of  $V_{\text{us}}$  from a lattice perspective. The recent progress are very impressive: computation with  $2+1$  and  $2+1+1$  dynamical flavours, physical pion mass, several fine lattices, different discretisation of the QCD Lagrangian, etc. (see for example the plenary talk given by Aida El-Khadra at this conference [1]). In this report, intended for non-lattice experts, I give an overview of the situation for the computation of  $f_K/f_\pi$  and  $f_+(0)$ , from which  $V_{\text{us}}$  and  $V_{\text{ud}}$  can be extracted. Besides the main features of the new computations, I also present some theoretical ideas developed in the recent years which allow for a cleaner determination of the relevant form factor  $f_+(0)$ . The experimental status has been reviewed in [2], and  $V_{\text{us}}$  from hadronic tau decay in [3].

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## 1 Introduction

The Cabibbo-Kobayashi-Maskawa (CKM) matrix describes quark flavour mixing in the Standard Model (SM). The unitarity relation imposes for the first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 . \quad (1.1)$$

The values given by the PDG 2012 read

$$V_{ud} = 0.97427(15), \quad V_{us} = 0.22534(65), \quad V_{ub} = 0.00351(15) . \quad (1.2)$$

With these numerical values, one clearly sees why finding a deviation of Eq.(1.1) is a difficult task, but with the constant improvement on both the experimental and the theoretical side, the first row is a very good framework for performing precise tests of the SM. We note that the value of  $|V_{ub}|^2$  is an order of magnitude smaller than the current uncertainty on the  $|V_{ud}|^2$  and  $|V_{us}|^2$ , which are of the same order.

There is currently a huge effort in the lattice community to improve the determination of  $V_{ud}$  and  $V_{us}$ . We refer the reader to FLAG [4] for a comprehensive review. In this proceeding, I present the recent ideas and highlight the newest computations.

## 2 Theoretical Framework - Lattice Computation

The basis idea is that since  $|V_{us}f_+(0)|$  and  $|V_{us}/V_{ud}|f_{K^\pm}/f_{\pi^\pm}$  are experimentally well measured (the numbers are taken from [4])

$$\begin{aligned} |V_{us}f_+(0)| &= 0.2163(5) \\ \left| \frac{V_{us}}{V_{ud}} \right| \frac{f_{K^\pm}}{f_{\pi^\pm}} &= 0.2758(5) , \end{aligned}$$

one can compute  $f_+(0)$  and  $f_{K^\pm}/f_{\pi^\pm}$  on the lattice and extract  $V_{us}$  and  $V_{ud}$ . (In this report we only consider QCD in the isospin limit  $m_u = m_d$ , and therefore do not write the charge explicitly, eg  $f_{p^+} = f_{p^-}$ , but electromagnetic corrections are applied [4].) We start with some basic definitions of the relevant form factors, first the decay constant

$$\langle 0 | A_\mu | P(p) \rangle = ip_\mu f_P , \quad \text{where } A_\mu = \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 . \quad (2.1)$$

Here  $P = \bar{\psi}_1 \gamma_5 \psi_2$  is either a pion or a kaon, hence Eq. (2.1) defines  $f_\pi$  and  $f_K$ . From the vector current  $V_\mu = \bar{\psi}_1 \gamma_\mu \psi_2$  we define the form factors  $f_+$  and  $f_-$

$$\langle \pi(p') | V_\mu | K(p) \rangle = (p + p')_\mu f_+(q^2) + (p - p')_\mu f_-(q^2) , \quad (2.2)$$

where  $q = p' - p$  is the momentum transfer. Finally we also introduce the scalar form factor  $f_0$  defined by ( $S = \bar{\psi}_1 \psi_2$ )

$$\langle \pi(p') | S | K(p) \rangle = \frac{m_K^2 - m_\pi^2}{m_s - m_l} f_0(q^2) . \quad (2.3)$$

The vector Ward Identity implies a relation between the vector current and the scalar density (for non-flavour singlet)  $\partial^\mu V_\mu = (m_2 - m_1)S$ . In particular, this gives

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2) . \quad (2.4)$$

In particular  $f_0(0) = f_+(0)$ , hence at zero-momentum transfer the form factor can either be extracted from the vector current, Eq. (2.2), or from the scalar density [5], see Eq. (2.3). A standard method introduced by [6] is to compute a ratio such as

$$\frac{\langle \pi | \bar{s} \gamma_0 l | K \rangle}{\langle \pi | \bar{l} \gamma_0 l | \pi \rangle} \frac{\langle K | \bar{l} \gamma_0 s | \pi \rangle}{\langle K | \bar{s} \gamma_0 s | K \rangle} = (f_0(q_{\max}^2))^2 \frac{(m_K + m_\pi)^2}{4m_K m_\pi} \quad (2.5)$$

where all the hadronic states are taken at rest and  $q_{\max}^2 = (m_K - m_\pi)^2$ . This ratio can be numerically very well determined (most of the systematics cancel out and the statistical precision is better at zero-momentum). In addition the same ratio can also be evaluated with non-vanishing momenta (for either the pion, the kaon or both) and the zero-momentum transfer form factor can be obtained by an interpolation (see for example [7,8]).

Simulating light quark masses is numerically expensive, and even if nowadays physical pion masses are accessible, one would like to take advantage of un-physical heavier quark since they are statistically more precise. The Ademollo-Gatto theorem plays a central role here: the form factor  $f_+(0)$  is exactly one in the  $SU(3)$  flavour limit and the first correction is parametrised by a known function  $f_2$ . In practise, one can use an Ansatz of the form:

$$f_+(0) = 1 + f_2(f, m_\pi^2, m_K^2) + \text{higher order} \quad (2.6)$$

Enormous progress have been made recently on the lattice side, development of new ideas, algorithms, discretisation of the Lagrangian, and of course hardware improvement, too numerous to be explained in detail in this report. Instead, I highlight some important improvements developed in the last years relevant for the lattice computation of  $V_{us}$

### *Theoretical developments*

- Thanks to partially twisted boundary conditions, the momenta are not restricted to the Fourier modes and the form factor can be computed directly a zero-momentum transfer [9]. No interpolation in momenta is required, avoiding a possible model-dependence Ansatz.
- The use of the scalar density (instead of the vector current) to extract  $f_0(0) = f_+(0)$ . One advantage is that in Eq (2.3) the quantity  $(m_2 - m_1)S$  is protected by a Ward Identity and hence no renormalisation is required.

### *Lattice improvements*

- Simulation with physical quark masses: FNAL/MILC and RBC-UKQCD are computing  $f_+(0)$  with light quarks down to their physical value [10,11]. FNAL/MILC simulates  $2+1+1$  dynamical flavours of Highly Improved Staggered Quarks (HISQ) and RBC-UKQCD simulates  $2+1$  flavours of Domain-Wall (DW) fermions, an action notoriously expensive which preserves chiral-flavour symmetry at finite lattice spacing. Hence the uncertainty due to the chiral extrapolation (which was the dominant one in 2013) is removed, or at least drastically reduced.

- Inclusion of dynamical quarks: in 2013, FLAG reported that three collaborations (FNAL/MILC, JLQCD and RBC/UKQCD) have computed  $f_+(0)$  with 2+1 flavours, ie a degenerate light doublet and a strange quark in the sea. More recently, two collaborations (ETM and the FNAL/MILC collaborations) have also included a dynamical charm: although one does not expect the charm to have a big effect in this sector, the lattice results are becoming so precise that this should certainly be checked. Let us also mention that HPQCD has computed  $f_K/f_\pi$  on the 2 + 1 + 1 MILC ensemble with physical quark masses [12].

### 3 Results: 2014 update

In 2013, the FLAG reported

$$f_+(0) = 0.9661(32) \quad n_f = 2 + 1 \quad (3.1)$$

$$f_+(0) = 0.9560(57)(62) \quad n_f = 2 \quad (3.2)$$

and noted that the major source of error came for the chiral extrapolation. We refer the reader to the original publications for more details [13,14,9,8,15,16].

These averages do not include the most recent results which are given in Table 1, together with some important features of the simulations. The action denotes the type of discretisation used for the Dirac operators. Even if the results should converge to the same continuum limit, at finite lattice spacing the theory suffers from distortion which are action-dependent. It is important to note that  $f_+(0)$  is now being computed with physical quark masses and that 2 + 1 + 1 results are also available.

Collaboration	Action	$m_\pi$ (MeV)	$a$ (fm)	$N_f$	$f_+(0)$	$ V_{us} $
FNAL/MILC [10]	HISQ	130	0.06	2 + 1 + 1	0.9704(32)	0.22290(74)(52)
ETM [17]	OS	210	0.06	2 + 1 + 1	0.9683(65)	0.2234(16)

**Table 1:** Summary of results for the most recent computations of  $f_+(0)$ , not included in the FLAG average yet. For each computation we give the lightest simulated pion mass, the finest lattice spacing and the number of quark flavours included in the sea. Note that the lightest pions mass is not necessarily simulated on the finest ensemble. The column “action” corresponds to the discretisation of the Dirac operator, see the original references for more details.

We now turn to the ratios of decay constant  $f_K/f_\pi$ . In their 2013 report, FLAG quoted

$$f_K/f_\pi = 1.194(5) \quad n_f = 2 + 1 + 1$$

$$f_K/f_\pi = 1.192(5) \quad n_f = 2 + 1$$

$$f_K/f_\pi = 1.205(6)(17) \quad n_f = 2$$

and again we refer the reader to the original work for more details [18,19,20,21,22,23,24,25,26,27,28]. Note that some of these results were obtained with physical quark masses. At

Collaboration	Action	$m_\pi$ (MeV)	$a$ (fm)	$N_f$	$f_K/f_\pi$
FNAL/MILC [29]	HISQ	130	0.06	2 + 1 + 1	1.1956(10) $\left(\begin{smallmatrix} +26 \\ -18 \end{smallmatrix}\right)$
RBC-UKQCD [30]	DW	139	0.08	2 + 1	1.1945(45)

**Table 2:** 2014 Update for  $f_K/f_\pi$ . The details are the same as in Table 1. The precision is to be compared to the FLAG13 average.

Lattice 2014, both the FNAL/MILC and the RBC-UKQCD collaborations have reported their new results, see Table 2.

It is interesting to look at the errors in more details. For example, for  $V_{us}/V_{ud}$  [29]

$$|V_{us}/V_{ud}| = 0.23081(52)_{\text{LQCD}}(29)_{\text{BR}(K_{12})}(21)_{\text{EM}} \quad \text{Fermilab Lattices/MILC 2014} \quad (3.3)$$

Even if the lattice errors still dominate, they are clearly becoming competitive

## 4 Conclusions - Outlook

The latest lattice simulations are truly impressive, tremendous progress have been made this year, in particular regarding the extraction of  $f_+(0)$ : the lattice simulations are now reaching the physical quark masses and include three or four flavour of dynamical quarks. The errors are usually dominated by the continuum extrapolation Ansatz. Improving this error by brute force (going to finer lattices) is a real challenge as it requires solving some theoretical issues (see for example [31]). Therefore it is very important to perform these computations with improved lattice actions (which have genuinely smaller lattice artifacts). Another challenge to face is that most of the lattice simulations are done in “pure” QCD in the isospin limit  $m_u = m_d$ . The results are now so precise that the effects of this approximation are becoming visible. For  $f_+(0)$  or  $f_K/f_\pi$ , a (model dependent) correction is applied *a posteriori* to the lattice results [4]. However, important progress have been made recently in that respect: see [32] for an implementation of QCD+QED at order  $\alpha$  and see [33] for a review.

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